MATHEMATICS

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MATHEMATICS
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INVERSE TRIGONOMETRIC FUNCTIONS

& Their Properties

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THINGS TO REMEMBER

✵ Inverse Functions :

If a function is one-one and onto from A to B then function g which associates each element $y \in B$ to one and only one element $x \in A$, such that $y = f(x)$, then g is called the inverse function of f, denoted by $x = g(y)$.

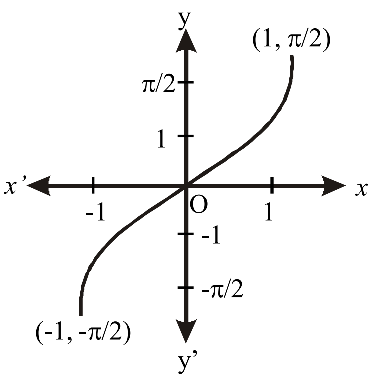
$\because \quad g = f^{-1}$

$\therefore \quad x = f^{-1}(y)$

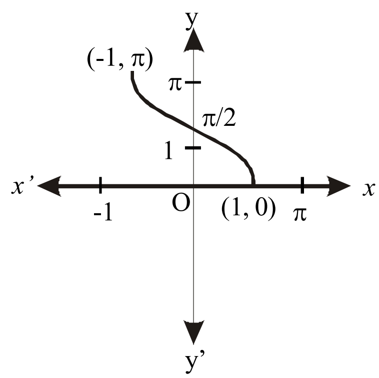
✵ Inverse Trigonometric Functions :

The trigonometric functions are periodic function, so they are not one to one and onto. But if we restrict their domains, then the inverse of trigonometric function exist.

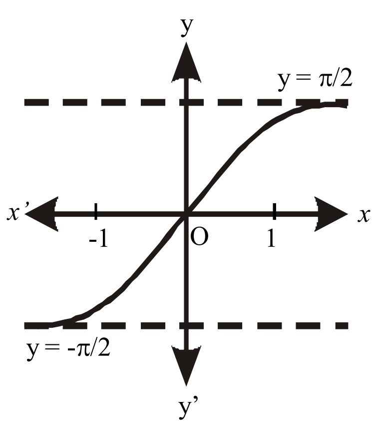
✵ Graph of Inverse Trigonometric Function :



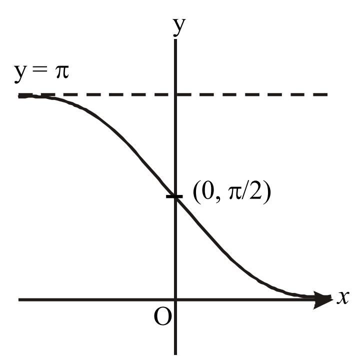
(a) $y = \sin^{-1}x$



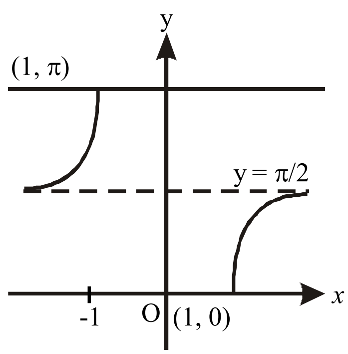
(b) $y = \cos^{-1}x$



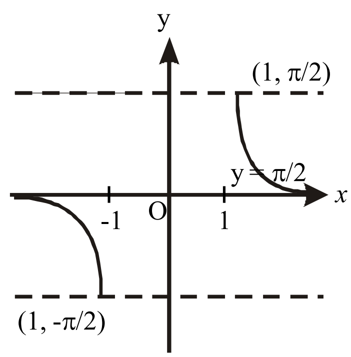
(c) $y = \tan^{-1}x$



(d) $y = \cot^{-1}x$



(e) $y = \sec^{-1}x$



(e) $y = \text{cosec}^{-1}x$

✷ Domain and Range of Inverse Trigonometric Functions :

S. No.	Function	Domain	Range
1.	$\sin^{-1}x$	[-1, 1]	[-π/2, π/2]
2.	$\cos^{-1}x$	[-1, 1]	[0, π]
3.	$\tan^{-1}x$	R	(-π/2, π/2)
4.	$\text{cosec}^{-1}x$	(-∞, -1] ∪ [1, ∞)	[-π/2, π/2] - {0}
5.	$\sec^{-1}x$	(-∞, -1] ∪ [1, ∞)	[0, π] - {π/2}
6.	$\cot^{-1}x$	R	(0, π)

✷ Inverse Trigonometric Functions :

S. No.	Function	Interval of Principal Value
1.	$\sin^{-1}x$	$-\pi/2 \le y \le \pi/2$, where $y = \sin^{-1} x$
2.	$\cos^{-1}x$	$0 \le y \le \pi$, where $y = \cos^{-1} x$
3.	$\tan^{-1}x$	$\pi/2 < y < \pi/2$, where $y = \tan^{-1} x$
4.	$\text{cosec}^{-1}x$	$-\pi/2 \le y \le \pi/2$, where $y = \text{cosec}^{-1}x$, $y \ne 0$
5.	$\sec^{-1}x$	$0 \le y \le \pi$, where $y = \sec^{-1}x$, $y \ne \pi/2$
6.	$\cot^{-1}x$	$0 < y < \pi$, where $y = \cot^{-1} x$

✷ Properties of Inverse Trigonometric Functions :

Self Adjustig Property

(i) $\sin^{-1}(\sin\theta) = \theta; \forall \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\cos^{-1}(\cos\theta) = \theta; \forall \theta \in [\theta, \pi]$

(iii) $\tan^{-1}(\tan\theta) = \theta; \forall \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(iv) $\cos ec^{-1}(\cos ec\,\theta) = \theta; \forall \theta \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \theta \ne 0$

(v) $\sec^{-1}(\sec\theta) = \theta; \forall \theta \in [0, \pi], \theta \ne \frac{\pi}{2}$

(vi) $\cot^{-1}(\cot\theta) = \theta; \forall \theta \in (0, \pi)$

(vii) $\sin(\sin^{-1} x) = x, \forall x \in [-1, 1]$

(viii) $\cos(\cos^{-1} x) = x, \forall x \in [-1, 1]$

(ix) $\tan(\tan^{-1} x) = x, \forall x \in R$

(x) $\operatorname{cosec}\left(\operatorname{cosec}^{-1} x\right)=x, \forall x \in(-\infty,-1] \cup[1, \infty)$

(xi) $\sec\left(\sec^{-1} x\right)=x, \forall x \in(-\infty,-1] \cup[1, \infty)$

(xii) $\cot\left(\cot^{-1} x\right)=x, \forall x \in R$

Negative Arguments

(i) $\sin^{-1}(-x) = -\sin^{-1}(x), \ \forall \ x \in [-1, 1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1}(x), \ \forall \ x \in [-1, 1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x, \ \forall \ x \in \mathrm{R}$

(iv) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x, \ \forall \ x \in (-\infty, -1] \cup [1, \infty]$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x, \ \forall \ x \in (-\infty, -1] \cup [1, \infty]$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1} x, \ \forall \ x \in \mathrm{R}$

Reciprocal Arguments

(i) $\sin^{-1}\left(\frac{1}{x}\right)=\operatorname{cosec}^{-1} x, \forall x \in(-\infty,-1] \cup[1, \infty)$

(ii) $\cos^{-1}\left(\frac{1}{x}\right)=\sec^{-1} x, \forall x \in(-\infty,-1] \cup[1, \infty)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right)=\begin{cases}\cot^{-1} x, & \forall x>0 \\ -\pi+\cot^{-1} x, & \forall x<0\end{cases}$

Inverse Sum Identities

(i) $\sin^{-1} x+\cos^{-1} x=\frac{\pi}{2}, \forall x \in[-1,1]$

(ii) $\tan^{-1} x+\cot^{-1} x=\frac{\pi}{2}, \forall x \in R$

(iii) $\sec^{-1} x+\operatorname{cosec}^{-1} x=\frac{\pi}{2}, \forall x \in(-\infty,-1] \cup[1, \infty)$

Sum and Difference of Inverse Trigonometric Function

(i) $\tan^{-1} x+\tan^{-1} y=\begin{cases}\tan^{-1}\left(\frac{x+y}{1-x y}\right), & \text{if } x y<1 \\ \pi+\tan^{-1}\left(\frac{x+y}{1-x y}\right), & \text{if } x>0, y>0 \text{ and } x y>1 \\ -\pi+\tan^{-1}\left(\frac{x+y}{1-x y}\right), & \text{if } x<0, y<0 \text{ and } x y>1\end{cases}$

(ii) $$\tan^{-1}x-\tan^{-1}y=\begin{cases}\tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & if\ xy>-1\\ \pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & if\ x>0, y<0\ and\ xy<-1\\ -\pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & if\ x<0, y>0\ and\ xy<-1\end{cases}$$

(iii) If $x_1, x_2, x_3, \ldots\ldots\ldots, x_n \in$ R, then

$$\tan^{-1}x_1+\tan^{-1}x_2+\ldots\ldots+\tan^{-1}x_n=\tan^{-1}\left(\frac{S_1-S_3+S_5-\ldots..}{1-S_2+S_4-S_6+\ldots..}\right)$$

Where, $S_1 = x_1 + x_2 + \ldots. + x_n = \Sigma x_1$

$S_2 = x_1x_2 + x_1x_3 + \ldots. + x_{n-1}x_n = \Sigma x_1x_2$

$S_2 = \Sigma x_1x_2x_3$.......... and so on.

(iv) $$\sin^{-1}x+\sin^{-1}y=\begin{cases}\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}, & \text{if } -1<x, y<1 \text{ and } x^2+y^2<1 \text{ or if } xy<0 \text{ and } x^2+y^2>1\\ \pi-\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}, & \text{if } 0<x, y<1 \text{ and } x^2+y^2>1\\ -\pi-\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}, & \text{if } -1<x, y<0, \text{ and } x^2+y^2>1\end{cases}$$

(v) $$\sin^{-1}x-\sin^{-1}y=\begin{cases}\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}, & \text{if } -1\le x, y\le 1 \text{ and } x^2+y^2\le 1 \text{ or if } xy>0 \text{ and } x^2+y^2>1\\ \pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}, & \text{if } 0<x\le 1,\ -1\le y\le 0 \text{ and } x^2+y^2>1\\ -\pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}, & \text{if } -1\le x<0,\ 0<y\le 1 \text{and } x^2+y^2>1\end{cases}$$

(vi) $$\cos^{-1}x+\cos^{-1}y=\begin{cases}\cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\}, & \text{if } -1\le x, y\le 1 \text{ and } x+y\ge 0\\ 2\pi-\cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\}, & \text{if } -1\le x, y\le 1 \text{ and } x+y\le 0\end{cases}$$

(vii) $$\cos^{-1}x-\cos^{-1}y=\begin{cases}\cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\}, & \text{if } -1\le x, y\le 1 \text{ and } x\le y\\ -\cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\}, & \text{if } -1\le y\le 0 \text{ and } 0<x\le 1 \text{ and } x\ge y\end{cases}$$

Conversion Property

(i) $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$

(ii) $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

(iii) $\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{1}{x}\right) = \sec^{-1}\sqrt{1-x^2} = \operatorname{cosec}^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

Inverse Trigonometric Ratio of Multiple Angles

(i)
$$2\sin^{-1} x = \begin{cases} \sin^{-1}\left(2x\sqrt{1-x^2}\right), & \text{if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right), & \text{if } -\frac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right), & \text{if } -1 \le x \le -\frac{1}{\sqrt{2}} \end{cases}$$

(ii)
$$3\sin^{-1} x = \begin{cases} \sin^{-1}\left(3x-4x^3\right), & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - \sin^{-1}\left(3x-4x^2\right), & \text{if } -\frac{1}{2} < x \le 1 \\ -\pi - \sin^{-1}\left(3x-4x^2\right), & \text{if } -1 \le x \le -\frac{1}{2} \end{cases}$$

(iii)
$$2\cos^{-1} x = \begin{cases} \cos^{-1}\left(2x^2-1\right), & \text{if } 0 \le x \le 1 \\ 2\pi - \cos^{-1}\left(2x^2-1\right), & \text{if } -1 \le x \le 0 \end{cases}$$

(iv)
$$3\cos^{-1} x = \begin{cases} \cos^{-1}\left(4x^3-3x\right), & \text{if } -\frac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}\left(4x^3-3x\right), & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 2\pi + \cos^{-1}\left(4x^3-3x\right), & \text{if } -1 \le x \le -\frac{1}{2} \end{cases}$$

(v)
$$2\tan^{-1} x = \begin{cases} \tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } -1 < x \le 1 \\ \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } x > 1 \\ -\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } x < -1 \end{cases}$$

(vi) $3\tan^{-1}x = \begin{cases} \tan^{-1}\left(\dfrac{3x-x^2}{1-3x^2}\right), & \text{if} -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\dfrac{3x-x^2}{1-3x^2}\right), & \text{if } x > \dfrac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\left(\dfrac{3x-x^2}{1-3x^2}\right), & \text{if } x < -\dfrac{1}{\sqrt{3}} \end{cases}$

(vii) $2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{if} -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{if } x > 1 \\ -\pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{if } x < -1 \end{cases}$

(viii) $2\tan^{-1}x = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & \text{if } 0 \le x \le \infty \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & \text{if } -\infty < x < 0 \end{cases}$

Note :

- If no branch of an inverse trigonometric function is mentioned, then it means the principal value branchof the function.
- If $\tan^{-1}x + \tan^{-1}y = \dfrac{\pi}{2}$, then $xy = 1$.
- If $\cot^{-1}x + \cot^{-1}y = \dfrac{\pi}{2}$, then $xy = 1$.
- if $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \dfrac{\pi}{2}$, then $x^2 + y^2 + z^2 + 2xyz = 1$
- if $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$ then $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$
- if $\sin^{-1}x + \sin^{-1}y = \theta$ then $\cos^{-1}x + \cos^{-1}y = \pi - \theta$
- if $\cos^{-1}x + \cos^{-1}y = \theta$ then $\sin^{-1}x + \sin^{-1}y = \pi - \theta$